Julia Robinson Mathematics Festival

Directed By Joshua Zucker

Squares and Sequences

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The Sequence

1. Beginning with 6, make a sequence of positive integers that is
   a) finite.
   b) Also increasing.
   c) Also has a product that is a perfect square.
   d) Make several more examples that fit the previous rules.
   e) Now find an example that has the least possible last number.

2. What happens if we begin with 9 instead? How about with 10?

3. Now let's make a table up to 30, where for each integer we find an increasing sequence of positive integers whose product is a perfect square and whose last number is as small as possible. There are only five rows given; it turns out that should actually be enough if you find the shortest sequence that works, at least up through 51.

   The last number of each little sequence forms “Ron Graham's Sequence”. (A006255)

4. What patterns do we find in Ron Graham's Sequence?
   a) What kinds of numbers are missing from our sequence ?
   b) What kinds of numbers are present in our sequence?
   c) How many times does each number occur?
   d) Can we be sure there aren't any missing?

Some Extra Problems

5. Create a sequence of positive integers following these rules:
   a) No number in the sequence is a perfect square.
   b) No product of any collection of adjacent numbers in the sequence is a square.
   c) At each step, choose the smallest integer possible.
   What do you notice about the numbers in the sequence? (A094290)

6. What sets of three consecutive numbers can all be written as the sum of two squares? We can see that one example is $72 = 6^2 + 6^2$, $73 = 8^2 + 3^2$, and $74 = 7^2 + 5^2$, but can you find another? Can you find a pattern that creates infinitely many of them? (A064716)

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1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

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21	22	23	24	25	26	27	28	29	30